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THE ROLE OF COST IN PRICING JOINT PRODUCTS:

A CASE OF PRODUCTION IN FIXED PROPORTIONS

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Introduction

The purpose of this paper is twofold: first, to examine the information requirements for the pricing of joint products arising in fixed proportions, and second, to identify the accounting procedures that will meet those requirements. The analysis is based on a deterministic model of the short-run price-output decision for two complementary joint products produced in fixed proportions. Consequently, the results of the analysis do not extend to cases of variable proportions or to other cases in which such a model is invalid. Moreover, the analysis focuses on the price-output decision and does not extend to problems of inventory valuation or income determination.

When cost is joint in relation to two outputs produced in fixed proportions, economists argue that any allocation of such cost between the products must be arbitrary unless determined by reference to demand functions. In other words,

an allocation lacks economic justification unless it reflects marginal revenues under the optimal price policy. Moreover, since the demand-based allocation of the joint cost derives from the optimal price policy, it cannot be used in reaching the optimal price policy. One might be tempted to conclude that accountants serve no economic purpose by assigning any joint cost to products produced in fixed proportions. Such a conclusion may be warranted when profit is maximized by selling the entire output after separate processing; in this case, the demand-based allocation must be derived from the optimal price policy and, hence, cannot be used in reaching that policy. But when profit maximization requires the sale or disposition of some production at split-off, then the demand-based allocation assigns the entire joint cost to the other product, all of which is separately processed. In this case the decision process may benefit by an accounting policy that resists assigning any joint cost to such production. More specifically, when demand information is incomplete such that split-off sales are known to be necessary but the optimal prices are unknown, then such an accounting policy may be useful in guiding experimentation with price-output possibilities in an effort to maximize profit. In this way, the economic analysis provides a limited justification for the by-product method of accounting for joint production.

When a joint product is marketed partly at split-off, the marginal cost of the separately processed portion is required

in order to determine its optimal price and sales level. Consequently, an accounting policy that makes that marginal cost available will facilitate profit maximization with respect to such production. Although economists and accountants[1] have recognized that the marginal cost of separately processed joint production (part of which is sold at split-off) is required in order to determine its profit maximizing price and sales level, the form of the required marginal cost has not been clearly specified for all cases of interest. The analysis below demonstrates that the appropriate form of the required marginal cost depends on the assumptions made concerning separate processing and disposition at split-off.

Formulation of the Maximization Problem

If two products are produced under fixed proportions in quantities x_1 and x_2, then the ratio of the outputs must be constant for all levels of production, that is, $x_1 = A \cdot x_2$. If the production process is subject to an upper limit on capacity, K, then two constraints are required to represent the relations of production. Let us suppose that x_1 may not exceed $(a_1 \cdot K)$ and that x_2 may not exceed $(a_2 \cdot K)$ where a_1 and a_2 give the production of the two products per unit of capacity. The upper limits on production of the two joint products may be written as follows:[2]

$$\begin{aligned} x_1 &= a_1 \cdot K \\ x_2 &= a_2 \cdot K \end{aligned} \tag{1}$$

In a typical cost accounting problem, we consider batch production in this way. K units of material are introduced to begin processing the batch and each unit of material input yields a_1 units of the first product and a_2 units of the second.

A distinction is made between (i) production of a joint product that is sold (or scrapped) at the split-off point and (ii) production of a joint product that is sold after further processing. If y_1 and y_2 represent separately processed joint production and u_1 and u_2 represent production sold (or scrapped) at the split-off point, then the capacity constraints may be written as follows:

$$\begin{aligned} y_1 + u_1 &= a_1 K \\ y_2 + u_2 &= a_2 K \end{aligned} \tag{2}$$

Provided that capacity, K, can be readily adjusted, only one of the two products will be sold at the split-off point; this result will be demonstrated below. A diagrammatic representation of these relationships and definitions is given in Figure 1. Unprocessed joint production is assumed to be sold in a perfectly competitive market at a constant unit contribution, s_i, given by the difference between a constant unit selling price and a constant unit selling cost.[3] The total contribution from sales of unprocessed production is given by:

$$s_1 u_1 + s_2 u_2 \tag{3}$$

where s_1 and s_2 are the constant unit contributions and u_1 and u_2 are the quantities of unprocessed joint production.

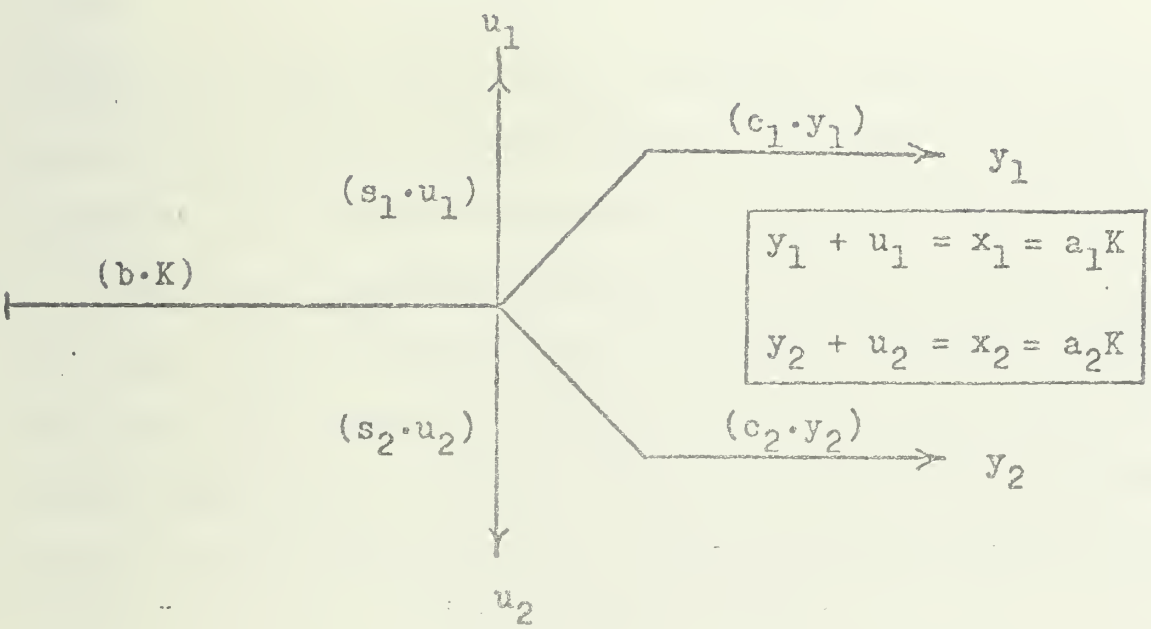

FIGURE 1.--Joint Production Problem

On the other hand, separately processed production is assumed to be sold in an imperfectly competitive market in which the quantity sold is inversely related to the unit price.[4] Since the objective of this analysis is to determine a profit-maximizing price policy, revenue from the separately processed joint production is written as a function of price as follows:[5]

$$p_1 \cdot D_1(p_1) + p_2 \cdot D_2(p_2) \tag{4}$$

where p_i is the price of the separately processed ith product and $D_i(p_i)$ is its demand function. In order to simplify the notation, the demand function will hereafter be written D_i.

The cost function recognizes two kinds of unit cost: (i) unit costs of processing the separate products beyond the split-off point, c_1 and c_2; and (ii) a unit cost of batch capacity, b. Since revenue is given as a function of price, cost is also given as the following function of price:[6]

$$c_1 D_1 + c_2 D_2 + b K \tag{5}$$

The first two terms are interpreted as the cost of processing the two products beyond the split-off point. The final term, bK, gives the cost which varies with the size of the batch, the capacity cost exemplified here by the cost of the jointly processed material input.[7]

By combining the revenue and cost functions given by equations (3), (4), and (5) and the joint production constraints given by equations (2), a general programming problem may be formulated as follows:

$$\underset{(p_1, p_2, u_1, u_2, K)}{\text{MAXIMIZE}} \quad \pi = p_1 D_1 + p_2 D_2 - c_1 D_1 - c_2 D_2 + s_1 u_1 + s_2 u_2 - bK$$

SUBJECT TO:

$$D_1 + u_1 = a_1 K$$
$$D_2 + u_2 = a_2 K$$
$$p_1,\ p_2,\ u_1,\ u_2,\ K \geqslant 0$$

The problem is to obtain the prices (p_1 and p_2), the quantities of unprocessed joint production (u_1 and u_2), and the batch size (K), that maximize profit on a production process issuing two joint products in fixed proportions where each product may be processed beyond the split-off point. Production subject to separate processing is sold in an imperfectly competitive market.[8] Production subject to disposition at the split-off point is sold in a perfectly competitive market to yield a unit contribution of s_i dollars for the ith product.

In order to derive necessary conditions for a solution to the maximization problem above, we write the Lagrangian form as follows:

$$L = \pi - \phi_1(D_1 + u_1 - a_1 K) - \phi_2(D_2 + u_2 - a_2 K) \tag{6}$$

where L is a function of p_1, p_2, u_1, u_2, and K. The dual

variables, ϕ_1 and ϕ_2, may be interpreted as the marginal opportunity costs of the two products. Moreover, the dual variables imply an assignment of joint cost between the products as will be shown below.

The necessary conditions[9] for a solution to the maximization problem are as follows:

(7) $p_1^* d_1^* + D_1^* - c_1 d_1^* - \phi_1 d_1^* \leqslant 0$

(8) $p_1^* (p_1^* d_1^* + D_1^* - c_1 d_1^* - \phi_1 d_1^*) = 0$

(9) $p_1^* \geqslant 0$

(10) $p_2^* d_2^* + D_2^* - c_2 d_2^* - \phi_2 d_2^* \leqslant 0$

(11) $p_2^* (p_2^* d_1^* + D_2^* - c_2 d_2^* - \phi_2 d_2^*) = 0$

(12) $p_2^* \geqslant 0$

(13) $s_1 - \phi_1 \leqslant 0$

(14) $u_1^* (s_1 - \phi_1) = 0$

(15) $u_1^* \geqslant 0$

(16) $s_2 - \phi_2 \leqslant 0$

(17) $u_2^* (s_2 - \phi_2) = 0$

(18) $u_2^* \geqslant 0$

(19) $- b + a_1 \phi_1 + a_2 \phi_2 \leqslant 0$

(20) $K^* (- b + a_1 \phi_1 + a_2 \phi_2) = 0$

(21) $K^* \geqslant 0$

Variables in the conditions above carrying a superscript asterisk (*) are optimal values, that is, the prices, disposal quantities and capacity that maximize profit. The symbol d_i is interpreted as the derivative of demand with respect to price, that is,

$$d_i = dD_i / dp_i,$$

and d_i^* is that derivative evaluated at the optimal price, p_i^*.

Optimal Price

Our primary purpose is to obtain a characterization of the optimal prices in terms of cost and demand parameters to enable recommendations for the assignment of costs. To this end, let us assume that price is strictly greater than zero; then conditions (8) and (11) yield the following expression for the optimal price:

$$p_i^* = -D_i^* / d_i^* + c_i + \phi_i, \quad i = 1,2. \tag{22}$$

This equation says that the optimal price must be the sum of the separate processing cost, c_i, and the marginal opportunity cost of the ith product, ϕ_i, plus the term $(-D_i^*/d_i^*)$ which derives from the demand function. Since D_i^* is positive and d_i^* is negative, the term $(-D_i^*/d_i^*)$ is positive. In other words, the optimal price is the sum of a positive demand component and the marginal cost $(c_i + \phi_i)$.

If we denote the price elasticity of the ith product by N_i where N_i is defined as $-d_i p_i/D_i$ (which is greater than zero provided d_i is negative) and if we let N_i^* be the elasticity evaluated at the optimal price, then equation (22) can be rewritten as follows:

$$(23) \qquad (N_i^* - 1)\, p_i^* = N_i^*\, (c_i + \phi_i), \quad i = 1,2.$$

Inspection of equation (23) shows (i) that $(c_i + \phi_i)$ is positive if and only if price elasticity, N_i^*, is strictly greater than unity; (ii) that $(c_i + \phi_i)$ is zero if and only if price elasticity equals unity; and (iii) that $(c_i + \phi_i)$ is negative if and only if price elasticity is strictly less than unity. This relationship between price elasticity and marginal cost is used to characterize the optimal output policy later in the paper.

In addition, marginal cost must equal marginal revenue at the optimal price. From the theory of demand, we know that price (p), marginal revenue (MR), and price elasticity (N) are related by the equation:

$$(24) \qquad (N - 1)\, p = N\, (MR)$$

where p is any price and demand is differentiable and has an inverse.[10] Since optimality requires that both (23) and (24) be satisfied, the optimal price, p_i^*, occurs at the equality of marginal revenue and marginal cost, that is, $MR = (c_i + \phi_i)$.

As noted above, the marginal opportunity costs, ϕ_1 and ϕ_2, imply an allocation of joint cost. This may be demonstrated by reference to equation (20). If batch capacity, K, is greater than zero, then condition (20) requires that the marginal capacity cost, b, equal the sum of the marginal opportunity costs weighted by the input-output coefficients, a_1 and a_2, that is,

$$b = a_1 \phi_1 + a_2 \phi_2 \tag{25}$$

This means that the unit capacity cost, b, must be completely allocated between the two joint products. In other words, ϕ_i may be interpreted as the share of b assigned to the ith product under the optimal pricing policy.

Further discussion of the implied allocation is facilitated by a distinction between two characterizations of the optimal solution which we shall call Case I and Case II. In Case I, the optimal price-output policy requires disposition or sale of production both at the split-off point and after separate processing.[11] In Case II, the optimal policy requires that the entire production of both products be marketed after separate processing.[12]

In general the marginal opportunity costs are found by solving the maximization problem. If Case II characterizes the optimal solution, then the marginal opportunity costs will be given in terms of both cost and demand parameters. But if Case I characterizes the solution, then the marginal opportunity costs are given in terms of cost parameters alone. In particular,

the entire joint processing cost is assigned to the product whose whole production is separately processed. Weil argues that the marginal opportunity costs lead to a proper and useful allocation of joint processing cost.[13] But how can they be useful in reaching the optimal price-output policy if they emerge simultaneously with it? Only if we know that Case I obtains but do not know the optimal prices will the marginal opportunity costs be useful in reaching the optimum. But, as Weil points out, the marginal opportunity costs may assist in reaching other decisions.[14]

Optimal Prices Under Case I

Pricing a Joint Product Marketed Both Before and After Separate Processing. If some of the first product is sold at the split-off point, a characterization of its marginal opportunity cost, ϕ_1, may be obtained from equation (14). If $u_1^* > 0$, then $\phi_1 = s_1$. In other words, when some of the first product is sold at split-off, the marginal opportunity cost is equal to the marginal contribution of sales at the split-off point. This means that expansion of the separately processed quantity reduces the quantity sold at the split-off point and thereby sacrifices contribution at a rate of s_1 dollars per unit.

Substituting s_1 for ϕ_1 in equation (22) yields the following expression for the optimal price of the separately processed production:

(26) $u_1^* > 0$ implies that $p_1^* = -D_1^*/d_1^* + c_1 + s_1$

that is, the price of separately processed production must provide for the recovery of both the unit cost of separate processing and the unit contribution foregone on sales at the split-off point.

The optimal price may be interpreted in terms of equation (23) to yield the relationship between the marginal cost $(c_1 + s_1)$ and the demand elasticity at which the separately processed production should be marketed. If the marginal cost is positive, then the separately processed production should be marketed at demand elasticity greater than unity. If, on the other hand, the unit contribution on sales at the split-off point is negative and sufficiently large in absolute value to make the marginal cost negative, then the separately processed production should be marketed at demand elasticity less than unity. In this case, the marginal cost is the cost of separately processing an additional unit less the disposition cost (plus the negative contribution) avoided by doing so. In the special case that the marginal cost equals zero, the separately processed production is marketed at the point of unitary demand elasticity.

These conclusions differ from those reached on the MS model, but they are not inconsistent with them. Manes and Smith conclude that "disposal of units of a product cannot yield maximum profit unless the product is marketed at the point of unitary demand elasticity, i.e., where marginal revenue = 0 (or where marginal revenue = marginal cost of processing and selling this item alone)."[15] The conclusion that sales of separately processed production should be expanded until marginal cost equals marginal revenue is correct. But marginal cost does not always equal zero and equality does not always occur at unitary demand elasticity. Consequently, revenue from separately processed production may be falling, rising or stable at the optimal price.[16]

In summary, when some production of a joint product is sold at the split-off point, the price of production sold after separate processing is given by the sum of (i) a positive demand component ($-D_i^*/d_i^*$), (ii) the unit cost of separate processing (c_i), and (iii) the unit contribution of sales at the split-off point (s_i). The separately processed production may be marketed at demand elasticity greater than, equal to, or less than unity depending on whether ($c_i + s_i$) is positive, zero, or negative.

<u>Pricing a Joint Product Marketed Only After Separate Processing</u>. We turn now to the price of the second product, the entire production of which is sold after separate processing. From equation (25) we know that the unit capacity cost, b, must

equal the sum of $a_1\phi_1$ and $a_2\phi_2$, where a_i is the number of units of the ith product produced by one unit of batch capacity and where ϕ_i' is the marginal opportunity cost of the ith product. Recall that if u_1^* is positive, then $\phi_1 = s_1$. Substituting this equality in equation (25) yields:

(27) $$\phi_2 = (1/a_2)(b - a_1 s_1)$$

This means that the marginal capacity cost of the second product equals the entire marginal capacity cost minus the marginal contribution on unprocessed production of the first product with respect to production of the second. This finding may be interpreted in terms of the price of the second product by substituting equation (27) in equation (22) as follows:

(28) $$p_2^* = - D_2^*/d_2^* + c_2 + b/a_2 - (a_1/a_2)\, s_1$$

The optimal price of the second product provides for the recovery of not only its marginal processing cost but also the entire capacity cost. In addition, the optimal price is reduced for a share of the contribution on unprocessed production of the first product. If the marginal cost of the second product ($c_2 + b/a_2 - a_1 s_1/a_2$) is positive, as is likely, then the product should be marketed at demand elasticity greater than unity.

Implementation of the Optimal Policy

The preceding sections examined the profit-maximizing price policy with special attention to the case in which some joint production is sold at split-off. The present section considers a graphical analysis of these results as well as results obtained by Manes and Smith and by Colberg under somewhat more restrictive assumptions. In general, the purpose of this section is to show how graphs may be used to obtain the profit-maximizing output policy from fully specified demand and cost functions.

Three production situations are examined. The first, which was examined by Manes and Smith, requires all joint production to receive separate processing and assumes that excess production receives costless disposition. The second, which was examined by Colberg, also assumes that excess production does not affect profit, but does not require that such production receive separate processing. The third situation accords with the model presented earlier in this paper and permits sales at split-off in a perfectly competitive market. Thus the third case comes closest to representing the joint cost problem as it is usually represented in cost accounting texts.

The optimal policy under the MS model may be demonstrated by reference to Figure 2. Since the two products arise in fixed proportions, both revenue and cost may be written as

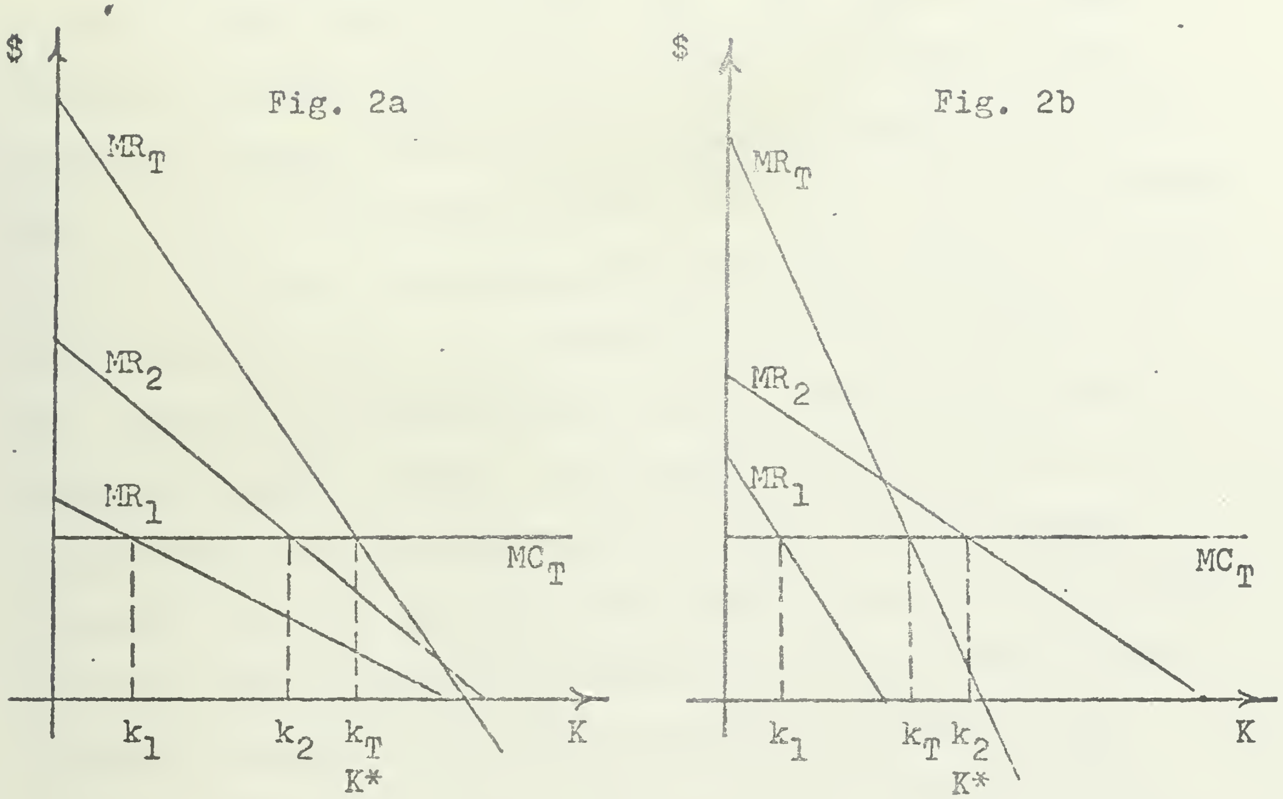
$
Fig. 2a
MR_T
MR_2
MR_1
MC_T
k_1
k_2
k_T
K*
K
$
Fig. 2b
MR_T
MR_2
MR_1
MC_T
k_1
k_T
k_2
K*
K

FIGURE 2

functions of K. Profit maximization requires that batch capacity, K, be set such that the appropriate marginal revenue equals total marginal cost. In Figure 2, total marginal revenue, MR_T, is the vertical summation of the marginal revenue lines of the separate products, MR_1 and MR_2. Since the MS model requires that all joint production receive separate processing, MC_T is $a_1c_1 + a_2c_2 + b$. All that needs be done is to determine the output levels, k_1, k_2, and k_T, at which marginal cost intersects the three marginal revenue lines. The optimal batch size, K*, is then simply the largest k-value. If k_T is the largest k-value, as it is in Figure 2a, then the entire production of both joint products is marketed at prices equal to the separate demand (or average revenue) functions at k_T. On the other hand, if k_2 is the largest k-value, as in Figure 2b, then the entire production of the second product is marketed at a price equal to average revenue at k_2, but only part of the first product's production will be marketed and that at the price corresponding to MR_1 of zero. The unmarketed production of the first product receives costless disposition. It is important to notice that the optimal policy was achieved without reference to any assignment of cost.

Colberg's model is like the MS model in that excess production of a joint product receives costless disposition, but it is unlike the MS model in that such production does not receive separate processi g. Again, three k-values are determined by reference to marginal cost as follows:

(i) k_1 is given by the intersection of MR_1 and $MC_1 = a_1c_1 + b$;

(ii) k_2 is given by the intersection of MR_2 and $MC_2 = a_2c_2 + b$;

(iii) k_T is given by the intersection of MR_T and $MC_T = a_1c_1 + a_2c_2 + b$.

As demonstrated in Figure 3, the optimal batch size, K^*, is given by the largest k-value. If k_T is the largest k-value, as in Figure 3a, then the entire production of both joint products in marketed at prices equal to the average revenue functions at k_T. On the other hand, if k_2 is the largest k-value, as in Figure 3b, then the entire production of the second product is marketed after separate processing at a price equal to average revenue at k_2. But only part of the first product's production will be marketed and that at the price corresponding to the equality of MR_1 and a_1c_1.

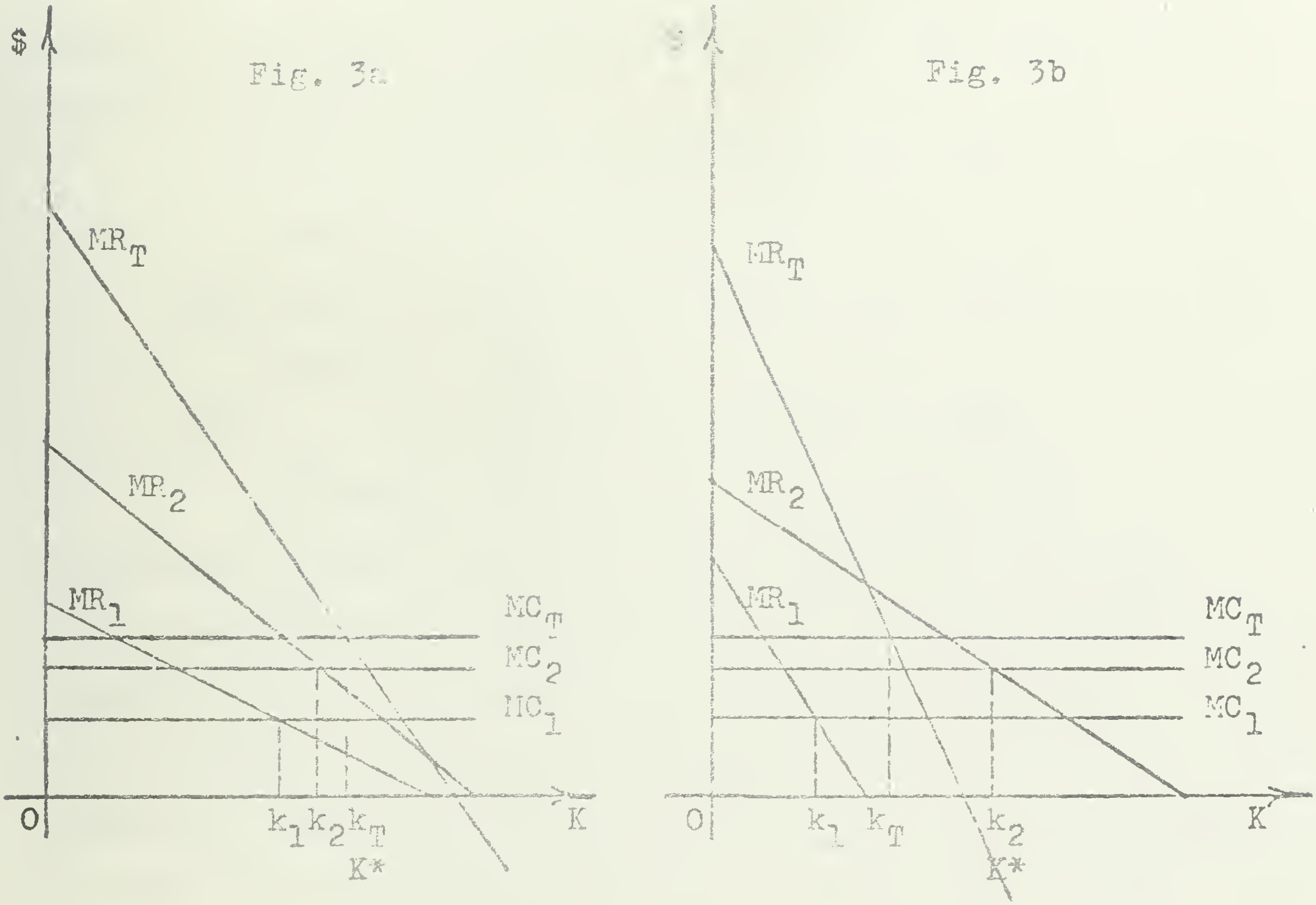
$
Fig. 3a
MR_T
MR_2
MR_1
MC_T
MC_2
MC_1
0
k_1 k_2 k_T
K*
K
Fig. 3b
MR_T
MR_2
MR_1
MC_T
MC_2
MC_1
0
k_1
k_T
k_2
K*
K

FIGURE 3

Under the model considered by this paper, the firm has the option of selling a joint product in a perfectly competitive market before it incurs the separate processing cost. Under this model, three k-values again lead to an identification of the optimal batch size. The three k-values are calculated as follows:

(i) k_1 is given by the intersection of MR_1 and $MC_1 = a_1c_1 - a_2s_2 + b$;

(ii) k_2 is given by the intersection of MR_2 and $MC_2 = a_2c_2 - a_1s_1 + b$;

(iii) k_T is given by the intersection of MR_T and $MC_T = (c_1 - s_1)a_1 + (c_2 - s_2)a_2 + b$.

Again, the largest k-value is the optimal batch capacity. If $k_T = K^*$, then the entire production of the two joint products is sold after separate processing (Case II). On the other hand, if k_2 is the largest k-value, as in Figure 3b, then the entire production of the second product is sold after separate processing. But only part of the first product's production will be marketed after separate processing; the remainder will be sold at split-off. In other words, an instance of Case I is indicated when either k_1 or k_2 exceeds k_T.

In summary, the optimal batch size under each model is the largest of three k-values--k_1, k_2, and k_T. k_1 is the optimal sales level for the first product when profit maximization requires sale or disposition of some of the second product at split-off. Similarly, k_2 is the optimal sales level for the second product when profit maximization requires sale or disposition of some of the first product at split-off. Finally, k_T is the optimal sales level for both products when profit maximization requires that the entire production of both products be sold after separate processing. While the three k-values have essentially the same interpretation under all three models, different marginal costs enter their determination in each model. But every marginal cost includes the marginal joint processing cost, b, which demonstrates that implementation of the optimal policy does not require an allocation of such cost under any of the models considered.

Accounting for By-Products

Manes and Smith argue that their analysis justifies the traditional approach to by-product cost accounting whenever profit maximization results in withholding some joint production from the market. They interpret the traditional approach "to allocate no input cost to the by-product and to indicate net revenue of the by-product either as a reduction in the cost of goods sold of the major product or as supplemental income."[17] The nature of the justification requires close scrutiny. Consider the implementation of the optimal policy under the Manes and Smith model. Suppose that k_2 is the maximum k-value; then part of the first product's output is withheld from the market. If the first product is treated as a by-product, it receives no cost; the entire input cost is assigned to the second product which is called the main or major product. But how does this assignment of cost assist in fixing the optimal prices? The optimal price of the second product is its average revenue at k_2, which is calculated without reference to cost. The optimal price of the first product is its average revenue at the sales volume that brings marginal revenue to zero where zero is the marginal cost of expanding sales of the first product. In other words, the marginal cost of the by-product is required in order to fix the optimal price and sales volume of the by-product. While the assignment of cost is not useful in fixing the optimal price of the main product, it is useful in fixing the optimal price of the by-product.

A similar argument can be made for the other models considered in the preceding section, but not without recommending a modification in the accounting procedure for by-products. The procedure must be modified to assign each unit of the separately processed by-product its marginal cost--(c_1) under the Colberg model and ($c_1 + s_1$) under the third model.[18]

Returning to the Manes and Smith model, there is a second way in which the marginal costs may be used in reaching the optimal policy. Suppose that demand functions are less than completely known; the firm knows only that some part of the first product's production must be scrapped as in Figure 2b.[19] How then does the firm maximize profit? In this case, it is useful to consider the separate marginal costs. The optimal prices of the two products are given as follows:

$$p_1^* = -D_1^*/d_1^*$$

$$p_2^* = -D_2^*/d_2^* + c_2 + (b + a_1c_1)/a_2$$

Since the positive demand components of the optimal prices are unknown, a reasonable strategy is to adjust the prices upward by small increments from the level of the marginal costs--zero for the first product and $c_2 + (b + a_1c_1)/a_2$ for the second--until incremental revenue equals incremental cost.

The firm would commence the process of adjustment by announcing a price for the second product equal to its marginal cost. Having satisfied the resulting demand,[20] the firm would repeat the process with an incremented price. The two observations of demand would enable the firm to compute marginal revenue. If the computed marginal revenue exceeds the marginal cost, the firm would announce a further increment in price and continue the process until marginal revenue fell to the level of marginal cost. Experimentation with the price of the first product (some of which is known to require costless disposition) would be conducted simultaneously in a similar way except that marginal revenue would be brought to zero.

The adjustment process for the other models considered in the preceding section would be identical except that the marginal costs would be calculated differently. In summary, when the firm's only knowledge of demand is that k_1 or k_2 will be the maximum k-value, then the marginal costs of the separate products may be useful in reaching the optimal price-output policy. In such cases, there is justification for an accounting procedure that assigns joint products their marginal cost.

Summary

The purpose of this paper is to examine the information requirements of the pricing decision for joint production in fixed proportions. When some production requires disposition or sale at split-off, the marginal opportunity costs, which imply an allocation of the joint processing cost, are given in terms of cost parameters alone. In that case, the joint product whose entire production is separately processed is allocated the entire joint cost. Such an allocation is useful in that the decision maker is not lead to consider the joint processing cost in setting the price and level of separate processing for the other product. The marginal cost of this product takes different forms depending on the conditions under which its sale at split-off and its separate processing take place; consequently, the accountant must exercise care in assigning the non-joint cost for purposes of the pricing decision. Furthermore, if Case I is known to obtain but demand information is incomplete (such that the optimal price-output policy is unknown), then the cost-based portion of the optimal prices may be useful in experimentation with price and output in an effort to reach the optimum.

FOOTNOTES

[1]See, for example: Marshal R. Colberg, "Monopoly Prices Under Joint Costs: Fixed Proportions," Journal of Political Economy, IL (1941), pp. 103-110; R. P. Manes and Vernon L. Smith, "Economic Joint Cost Theory and Accounting Practice," The Accounting Review, XL (January, 1965), pp. 31-51; and Roman L. Weil, Jr., "Allocating Joint Costs," American Economic Review, LVIII (December, 1968), pp. 1342-1345.

[2]The analysis assumes that capacity, K, and production, x_1 and x_2, are perfectly divisible.

[3]This assumption is less restrictive than those employed by Colberg (1941) and Manes and Smith (1965). Manes and Smith assume that all production is either sold after separate processing or scrapped without additional cost and that all production is separately processed. Colberg's most general model does not require that all production be separately processed but does require costless disposition at split-off. The model considered here is somewhat more general in that all production need not be separately processed and in that unprocessed production may generate a nonzero contribution at split-off. All three models, however, are single-period analyses under which production for inventory is precluded; all production is marketed or scrapped within the production period.

[4]This characterizes both the Manes and Smith model and the models considered by Colberg.

[5]Revenue from separately processed production of the ith joint product, R_i, may be written in two ways. In general, revenue is the product of price and quantity sold, but that product may be given either as a function of price, p_i, or as a function of quantity sold, y_i, that is,

$$\begin{aligned} R_i &= p_i \cdot y_i \\ &= p_i \cdot D_i(p_i) \\ &= Z_i(y_i) \cdot y_i \end{aligned}$$

where $y_i = D_i(p_i)$, which is the demand function, and $p_i = Z_i(y_i)$, which is the inverse demand function. If the inverse exists, then the two formulations of revenue are equivalent, and an optimal price is equivalent to an optimal sales level. The equations above assume that the demands for the two products are independent of one another; hence, the demand for each product is a function of its price alone and not the price of the other product. Similarly, the price of each product is a function of its sales alone and not the sales of the other product.

[6] If the solution to the maximization problem were sought in terms of outputs rather than prices, then the cost function would have the following form:

$$c_1 \cdot y_1 + c_2 \cdot y_2 + b \cdot K$$

[7] A production process may generate a joint product (e.g., scrap) that is not susceptible of further processing. If such production is sold in a perfectly competitive market, then its contribution per unit of capacity merely reduces the unit capacity cost, b. This accords with the recommended procedure of crediting the net realizable value of scrap to the cost of joint production. This type of joint product is distinguished from unprocessed joint production that is susceptible of further processing.

[8] If all production is sold in perfectly competitive markets, then the problem reduces to a linear programming problem in which prices are known constants. A thorough treatment of problems of this type is given by Ronald V. Hartley, "Decision Making When Joint Products Are Involved," The Accounting Review, XLVI (October, 1971), 746-755.

[9] The necessary conditions for a saddle-point solution are also sufficient provided that profit is a concave function of the prices, disposal quantities and capacity. Profit is concave if and only if D_1 and D_2 are concave when $p_1 - c_1 \geqslant 0$ and $p_2 - c_2 \geqslant 0$, which in turn is true if and only if the second derivative of demands with respect to price, d_1^2 and d_2^2, are zero or negative. In short, if we are willing to assume that the second derivative of both demand functions will not assume positive values, then the conditions above are both necessary and sufficient for p_i when $p_i - c_i \geqslant 0$. Since the second derivative of a linear demand function is zero, the requirement is fulfilled for our example. See G. Hadley, Nonlinear and Dynamic Programming (Reading, Massachusetts: Addison-Wesley Publishing Company, Inc., 1964).

[10] See C. E. Ferguson, Microeconomic Theory (Homewood, Illinois: Richard D. Irwin, Inc., 1969), pp. 100-102.

[11] If part of one product's production is sold at the split-off point, then the entire production of the other product will almost always be marketed after separate processing. This result assumes that K, the batch capacity, is sufficiently adjustable to be set at the optimal level. If u_1^* is positive, then $\phi_1 = s_1$. From (20), $\phi_2 = b/a_2 - (a_1/a_2)s_1$ provided k is positive. Substituting in (17), we have the requirement that

$$u_2^* \left(s_2 - b/a_2 + a_1 s_2/a_2\right) = 0$$

Since the term in the parentheses rarely equals zero, u_2^* nearly always equals zero. If u_2^* equals zero, then the entire production of the second product is sold after separate processing.

[12] A third case might be identified in which the entire production is sold at split-off. In this case, no price policy is necessary because the markets for production at split-off are perfectly competitive.

[13] Weil (1968), pp. 1342-1343.

[14] Ibid., p. 1343.

[15] Manes and Smith (1965), p. 33.

[16] The difference in conclusions is a consequence of their assumption that marginal cost of units scrapped at split-off equals the marginal cost of units sold after separate processing or, equivalently, that excess production receives costless disposition after incurring the separate processing cost. This assumption enables them to write the optimal price in terms of demand parameters alone; from this and the assumed positive price, it follows that demand elasticity must be unity. This result can be demonstrated under the present model. If we let $s_1 = - c_1$, when $u_1^* > 0$, then $\phi_1 = - c_1$ and the right-hand side of equation (23) is zero. Since price is positive, this means that N_1^* must equal unity. Moreover, from equation (22), price is given by the demand component alone.

[17]Manes and Smith (1965), p. 33.

[18]It is important to notice that Case I does not coincide with the by-product case as usually defined. Cost accounting texts define a by-product as a joint product whose importance or value is small relative to other joint products from the same process. Although the definition is not precise, it is clear that it may be interpreted to include instances of Case II and to exclude instances of Case I. Moreover, the definition is not restricted to fixed proportions. Consequently, when we say that the analysis of Case I justifies by-product accounting we refer only to the coincidence of the definition and Case I.

[19] In the absence of complete specification of demand functions, the separate marginal costs are known when either k_1 or k_2 is known to be the largest k-value. But the marginal costs of the separate products are not known when k_T is expected to be the largest k-value unless the demand functions are also known. Consequently, if demand functions are unknown, except to the extent that k_T is expected to be the largest k-value, separate marginal costs cannot be used to pursue the optimal policy.

[20] Careful judgment must be exercised in this process of adjustment. If, for example, the firm is unable to satisfy demand in any period, demand in the following period may be affected and some adjustment would be in order.